Physics Motivation

- One can model magnetism by assigning a vector $\vec{J}$ (spin) for every point on a surface

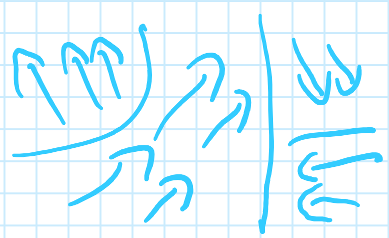

Fig 1: $T > T_c$

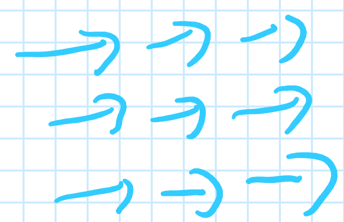

Fig 2: $T < T_c$

- Let T = temperature. $\exists$ temp T_c "Curie temp" s.t. $T < T_c$ the surface is magnetized (spins aligned), $T > T_c$ —— has no magnetic field (spins random) and $T = T_c$ there is a phase transition:

Q: How to find T_c?

A: Compute the partition function Z

Def A stat mech system Π is a collection of states $\{\theta\} = (\vec{\theta}_x)_{x \in P}$, $\vec{\theta}_x \in \mathbb{R}^d$, P = sites with a Hamiltonian H : States $\longrightarrow \mathbb{R}$ (energy)

Ex: $P = 3 \times 3$ grid, $\vec{\theta}_x = (1,0)$ or $(-1,0)$ $|\{\theta\}| = 2^9$

Def Given Π, the partition function is

$$Z_{\Pi}(T) = \sum_{\{\theta\}} e^{\frac{-H(\{\theta\})}{T}}$$

- Given a state $\{\theta\}$, the probability of $\{\theta\}$ is

$$P(\{\theta\}) = \frac{e^{-H(\{\theta\})/T}}{Z_{\Pi}(T)}$$

Rmrk (1) $H(\{\theta\})$ (energy) $\uparrow \longleftrightarrow P(\{\theta\}) \downarrow$

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(2) $T \to \infty \Rightarrow P(\{\theta\})$ ind of $\{\theta\} \Rightarrow \{\theta\}$ will have uniform (random) distribution

$T \to 0 \Rightarrow$ The state $\{\theta\}$ w/ smallest energy $H(\theta)$ will dominate (spins aligned)

- Any function $O: \text{states} \to \mathbb{R}$ is called an observable

Def: Given an observable O, the expectation value of O is

$$\langle O\rangle(T) = \frac{\sum O(\{\theta\})e^{-H(\{\theta\})/T}}{Z_\pi(T)}$$

Ex: $H(\{\theta\})$ is an observable

Def $U(T) = \langle H\rangle(T)$ is the internal energy of π

Def The free energy of π is

$$F(T) = -T\ln Z_\pi(T)$$

Def The entropy of π is $S(T) = \frac{\partial}{\partial T}F(T)$

- Note how $U(T), F(T), S(T)$ are all derived from $Z_\pi(T)$ ⇝ computing $Z_\pi(T)$ is of fundamental importance in stat mech

Def: A stat mech system π is called exactly solved if $Z_\pi(T)$ has a closed form, i.e. can be computed.

Def The 2-point correlation function of $x, y \in P$ is

$$G^{(2)}(x,y) = \langle \vec{\theta_x} \cdot \vec{\theta_y} \rangle$$ dot product

Rem:

Wick rotation

Stat Mech	QFT
• $Z_\pi(T)$	• Feynman path integral
• n-point correlator of sites x	• n-point correlator of fields $\phi(x)$

Informal Def A vector space V over F is a set of "vectors" $\vec{v} \in V$ s.t.

1. You can add vectors: $\vec{v} + \vec{w} \in V$
2. You can scale vectors by elements of F: $r\vec{v} \in V$ $\forall r \in F$

satisfying axioms (Ex: can subtract vectors)

Def A subset $B = \{\vec{v}_1, \dots, \vec{v}_m\} \subseteq V$ (f.d.) is a basis if

(1) if $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = 0$ for some $c_1, \dots, c_m \in F$ then $c_1 = \dots = c_m = 0$ (L.I.)

(2) $\forall \vec{v} \in V$, $\exists$ $a_1, \dots, a_n \in F$, $\exists$ $\vec{v}_1, \dots, \vec{v}_n \in B$ s.t. $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ (spanning)

Ex: $V = \mathbb{R}^n$, $F = \mathbb{R}$ then $B = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_2}, \dots, \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}}_{\vec{e}_n} \right\}$ is a basis for V

$(c_1, \dots, c_n) = c_1\vec{e}_1 + \dots + c_n\vec{e}_n$

Def The dual vector space V^* of V is

$$V^* = \{ f : V \to F \mid f \text{ is linear} \}$$

- $f_1 + f_2$ is still linear
- rf is still ~~lin~~ fine

Ex: Let $V = \mathbb{R}^n$

$$e_j^*(e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

and extend by linearity, e.g.

$$e_j^*(c_1e_1 + c_2e_2) = c_1e_j^*(e_1) + c_2e_j^*(e_2)$$

$\rightsquigarrow$ this forces e_j^* to be linear

- Can generalize to any basis of V

Def Given a basis $\{v_j\}_{j=1}^n$ of V, $\{v_j^*\}$ is a basis for V^*, called the dual basis where

$$v_j^*(v_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

and extend by linearity

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Remark: Let $\langle , \rangle : V \times V \to \mathbb{R}$ be the std inner prod on $V = \mathbb{R}^n$, aka dot product

$$\langle (c_1, \dots, c_n), (a_1, \dots, a_n) \rangle = c_1 a_1 + \dots + c_n a_n$$

Then $e_j^*(-) = \langle e_j, - \rangle$ as $\langle e_j, e_i \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$ and $\langle e_j, - \rangle$ is linear

Warning: Physicists use bra-ket notation

$$v_j^* = \langle v_j | , \quad v_i = | v_i \rangle$$

And $\langle v_j | v_i \rangle \neq \langle v_j, v_i \rangle$ unless $\{v_j\}$ is an orthonormal basis for V

Ex: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$\langle v_1 | v_2 \rangle = 0 \neq 1 = \langle v_1, v_2 \rangle$$

Lemma 1: By fixing a basis $B_n = \{\vec{v_1}, \dots, \vec{v_n}\}$ of $\mathbb{R}^n$ and a basis $B_m = \{w_1, \dots, w_m\}$ of $\mathbb{R}^m$

$$\left\{ \begin{matrix} \text{linear maps} \\ \mathbb{R}^n \to \mathbb{R}^m \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} m \times n \text{ matrices} \\ M_{m \times n}(\mathbb{R}) \end{matrix} \right\}$$

Pf:

$$A \longleftrightarrow M_A = \begin{array}{c} \begin{matrix} A(v_1) & \dots & A(v_n) \end{matrix} \\ \begin{pmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix} \end{array}, \quad M_A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} = A \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$A(\vec{v_1}) = a_{11} \vec{w_1} + \dots + a_{m1} \vec{w_m}$$

$$\vdots$$

$$A(v_n) = a_{1n} \vec{w_1} + \dots + a_{mn} \vec{w_n}$$

Def: $A(w_i, v_j) := a_{ij}$ are the matrix elements of A

Lemma 2: $A(w_i, v_j) = \langle w_i | A | v_j \rangle$

Pf: $\langle w_i | A | v_j \rangle = \langle w_i | a_{1j} \vec{w_1} + \dots + a_{nj} \vec{w_n} \rangle$

$$= a_{ij}$$

Def An eigenvector for A is a vector $\vec{v}$ s.t. $A\vec{v} = \lambda\vec{v}$, $\lambda \in F$ is the corr eigenvalue

Def: A matrix A is diagonalizable if $\exists$ invertible matrix S s.t. $A = S^{-1}DS$ where D is a diagonal matrix $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Prop: $Tr(AB) = Tr(BA)$

Pf: $AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & & \ddots & \\ \vdots & & & \\ b_{n1} & & & b_{nn} \end{pmatrix}$

$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & & \\ & \ddots & \\ & & a_{n1}b_{1n} + \cdots + a_{nn}b_{nn} \end{pmatrix}$

$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} & & \\ & \ddots & \\ & & b_{n1}a_{1n} + \cdots + b_{nn}a_{nn} \end{pmatrix}$

Thrm: If A is diagonalizable then

$$Tr(A) = \sum_{i=1}^{n} \lambda_i$$ ← eigenvalues of A counted w/ multiplicity

Pf: A diag $\Rightarrow A = SDS^{-1}$

Claim 1: e.values of A = e.values of D.

Pf: Let v s.t. $Av = \lambda v \Leftrightarrow (A - \lambda I)(v) = 0$

$\Leftrightarrow (SDS^{-1} - \lambda I)(v) = 0 \overset{\text{apply } S^{-1} + S^{-1}(0)=0}{\Longleftrightarrow} (DS^{-1} - S^{-1}\lambda I)(v) = 0$

$\overset{S \text{ invertible}}{\Longleftrightarrow} (DS^{-1} - S^{-1}\lambda I)(Sw) \overset{\lambda I S = S \lambda I}{\Longleftrightarrow} (D - \lambda I)(w) = 0$

$Tr(A) = Tr((SD)S^{-1}) \overset{\text{Prop}}{=} Tr(S^{-1}(SD)) = Tr$

– but eigenvalues + eigenvectors of D very easy to find

$e_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{matrix} \text{first} \\ \text{col} \\ \text{of } D \end{matrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\rightsquigarrow$ eigenvalues of D are the diagonal entries (w/ eigenvectors $e_1, \ldots, e_n$) $\rightarrow Tr(D) = \sum_{i=1}^{n} \lambda_i$

λ_i e.vectors of $D \overset{\text{Claim 1}}{=\!=} $ e.vectors of A

Def Two matrices A, B are simultaneously diagonalizable if $\exists$ invertible matrix S s.t. $A = S^{-1} D_1 S$, $B = S^{-1} D_2 S$, D_i diagonal

$\Leftrightarrow$ A, B have the same eigenvectors

Thrm: If A, B are diagonalizable and $[A,B] = AB - BA = 0$, then A, B are simultaneously diagonalizable

Ex:
- $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ commute
- $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are $e\vec{v}$ for $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Check:

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+4 \\ -4+3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus for $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Warning; The eigenvalues of simul diagonalizable matrices might be different!

Thrm: If $A_1, \dots, A_k$ are pairwise commuting lin operators on V, and each A_i d-zable then they are simul d-zable at the same time.